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A NOVEL INTEGRAL EQUATION APPROACH FOR WAVE PROPAGATION IN INHOMOGENEOUS DIELECTRIC SLABS

LOTFI RABEH GOMAA
Faculty of engineering of Shobra, Electrical Engineering
Department, Zaqaziq university, Banha branch - Egypt

Abstract

I derive an integral equation using Abel's method to solve the Helmholtz equation in a one-dimensional inhomogeneous dielectric slab. The kernel of the integral equation is separable and is of a non-convolution type. A simple direct iterative technique to solve that equation is presented. The reflection and transmission coefficients of some previously studied profiles are considered. A comparison between the results of this formulation and those of two other methods shows good agreement.

I-Introduction

Wave propagation in one-dimensional inhomogeneous dielectric slab is important for many practical reasons. The old problem of ionospheric propagation has long been a subject of study[1]. The transmission and reflection properties of plasma sheaths[2], stratified media[3],[4], and the layered troposphere[5] have been studied extensively. Several methods have been developed for analyzing wave propagation in one-dimensional inhomogeneous media[6]-[14]. Only a few refractive index profiles yield closed-form analytic solutions[1],[3],[4]. For other profiles, only approximate methods of solution are possible: these include the WKB method[6], profile discretization[4], Hill's function[7], Riccati's differential equation[8], Chandrasekhar transformation[9], Bremmer series[10], finite element method[11], integral equations[12],[13] and perturbation technique[14].

The integral equation formulation is one of the most powerful methods for studying wave propagation in inhomogeneous slabs, but the numerical techniques that can be used to solve these integral equations are very limited. Wang[13] and Hassab[14] derived two integral equations but, as pointed out by Chen[12], their equations are not appropriate for numerical computations. Chen[12] used the Green's function and the induced current concept to derive an integral equation for the field in the inhomogeneous slab. Then, to solve his equation numerically, he used a quadratic zoning function as an approximation to the field inside the slab; this transforms the integral equation into a system of algebraic equations which can be solved by matrix manipulations.

To the author's knowledge, there does not exist any published integral equation formulation appropriate for "direct" numerical computations (i.e., in which the integral equation is not converted to any other type of equation). In this paper we present a formulation of the problem based on Abel's method[15], so that the resulting integral equation is solvable directly using a simple iterative technique. Our numerical results are compared with those obtained by two other methods: Chen's integral equation formulation[12], and the finite element method[11].

II- Formulation of the integral equation

Consider a one-dimensional lossless inhomogeneous dielectric slab which occupies the space $0 \le x \le a$. The spaces $x<0$ and $x>a$ are filled with two lossless homogeneous dielectric media whose relative permittivities are ε_i and ε_t respectively as shown in figure 1.

Assume a linearly polarized plane wave $\hat{y}\exp(jk_i x)$ to be incident normally onto the slab from the left side (where $\hat{y}$ is a unit vector in the y-direction and k_i is the wavenumber in the space $x<0$).

The time dependence $\exp(-j\omega t)$ is assumed. The reflected and transmitted fields in the spaces $x\leq 0$ and $x>a$ can be obtained if the reflection and transmission coefficients R and T are known. These coefficients can be calculated if the electric field $\vec{E}=\hat{y}\phi(x)$ in the inhomogeneous slab is known. In the space $x<0$ the field $\phi_i(x)$ satisfies the Helmholtz equation:

$$\frac{d^2}{dx^2}\phi_i + k_o^2\,\varepsilon_i\,\phi_i = 0 \quad \text{------------------------(1)}$$

where k_o is the free space wavenumber.

The solution of (1) in the region $x<0$ is simply [12]:

$$\phi_i(x) = \exp(jk_i x) + R\exp(-jk_i x) \quad \text{------------------------(2)}$$

where R denotes the reflection coefficient.

Similarly, the field $\phi_t(x)$ in the region $x>a$ can be written as [12]:

$$\phi_t(x) = T\exp[jk_t(x-a)] \quad ,x>a \quad \text{------------------------(3)}$$

where k_t is the wavenumber in the space $x>a$ and T is the transmission coefficient. The field $\phi(x)$ in the inhomogeneous slab satisfies the differential equation [13]:

$$\frac{d^2}{dx^2}\phi + k_o^2\varepsilon(x)\phi(x) = 0 \quad \text{------------------------(4)}$$

Let us write the relative permittivity distribution $\varepsilon(x)$ as the sum of two parts:

$$\varepsilon(x) = \varepsilon_a - \delta\varepsilon(x) \quad \text{------------------------(5)}$$

One of these parts ε_a is a constant and the other part $\delta\varepsilon(x)$ can be considered as a continuous perturbation [14] on ε_a. The constant part ε_a can be taken, for example, as the average value of $\varepsilon(x)$, i.e.:

$$\varepsilon_a = \frac{1}{a}\int_0^a \varepsilon(x)dx$$

Accordingly, (4) can be written as the inhomogeneous differential equation:

$$\frac{d^2}{dx^2}\phi + k_o^2\varepsilon_a\phi(x) = k_o^2\delta\varepsilon(x)\phi(x) \quad (6)$$

which can be solved using Abel's method[15]. The solution of (6) can be expressed in terms of the solutions of the corresponding homogeneous equation:

$$\frac{d^2}{dx^2}\phi + k_o^2\varepsilon_a\phi(x) = 0 \quad (7)$$

It is worthwhile to note that (7) means that the original problem is reduced to a homogeneous slab sandwiched between two homogeneous dielectrics ε_i and ε_t occupying the spaces x<0 and x>a respectively, i.e. the relative permittivity profile in the space $-\infty<x<\infty$ is:

$$\varepsilon = \begin{cases} \varepsilon_i & \text{for } x<0 \\ \varepsilon_a & \text{for } 0\le x\le a \\ \varepsilon_t & \text{for } x>a \end{cases} \quad (8)$$

The solution of (7) in $0\le x\le a$ when a unit amplitude plane wave is incident from x<0 is elementary[16]:

$$\phi(x) = \phi_1(x) + \phi_2(x) \quad (9)$$

where:

$$\phi_1(x) = E^+\exp(jk_a x) \quad (10)$$

and

$$\phi_2(x) = E^-\exp(-jk_a x) \quad (11)$$

where k_a is the wavenumber in $0\le x\le a$ for the profile defined by equation (8). The constants E^+ and E^- are primitive and can be found by matching the tangential components of the field at the interfaces x=0 and x=a, they are given by:

$$E^{+} = \frac{2(1+n_{ta})}{[(1+n_{ai})(1+n_{ta})] + [(1-n_{ai})(1-n_{ta})\cdot\exp(2jk_a a)]} \quad \text{-------(11a)}$$

$$E^{-} = 1 + Y - E^{+} \quad \text{------------------------------------(11b)}$$

where:

$$Y = \frac{[(1+n_{ta})(1-n_{ai})] + [(1-n_{ta})(1+n_{ai})\cdot\exp(2jk_a a)]}{[(1+n_{ai})(1+n_{ta})] + [(1-n_{ai})(1-n_{ta})\cdot\exp(2jk_a a)]} \quad \text{-----(11c)}$$

and $n_{ai} = n_a/n_i$, $n_{ta} = n_t/n_a$, where n_i, n_a and n_t are the refractive indecies of the three regions $x<0$, $0\leq x\leq a$ and $x>a$ respectively. Using Abel's method[15], we can write the general solution of (6) for the field $\phi(x)$ in the inhomogeneous slab as follows:

$$\phi(x) = A\phi_1(x) + B\phi_2(x) + \frac{1}{W}\int_0^x G(x,\xi)\, k_o^2\, \delta\varepsilon(\xi)\, \phi(\xi)\, d\xi \quad \text{---------(12)}$$

where A and B are constants to be determined from the boundary conditions and the kernel $G(x,\xi)$ is given by[15]:

$$G(x,\xi) = \phi_1(\xi)\,\phi_2(x) - \phi_1(x)\,\phi_2(\xi) \quad \text{---------------------------(13)}$$

The constant W is the wronskian:

$$W = \phi_1\phi_2' - \phi_1'\phi_2 = -2jk_o\sqrt{\varepsilon_a}\, E^{+}E^{-} \quad \text{------------------------(14)}$$

where the primes denote differentiation with respect to x. The four unknowns A, B, R and T in the equations (2), (3) and (12) are found from the continuity of $\phi(x)$ and its derivative at the boundaries x=0 and x=a. At x=0, the boundary conditions are:

$$1 + R = A\,\phi_1(0) + B\,\phi_2(0) \quad \text{--(15)}$$

$$jk_i - jk_i R = A\,\phi_1'(0) + B\,\phi_2'(0) \quad \text{-------------------------------------(16)}$$

he continuity of $\phi(x)$ and its derivative at x=a gives:

$$A\phi_1(a) + B\phi_2(a) + \phi_2(a)I_1 - \phi_1(a)I_2 = T \quad \text{------------------------(17)}$$

$$A\phi_1'(a) + B\phi_2'(a) + \phi_2'(a)I_1 - \phi_1'(a)I_2 = jTk_t \quad \text{------------------------(18)}$$

here $I_1 = \frac{1}{W}\int_0^a \phi_1(\xi)\, k_o^2 \delta\varepsilon(\xi)\, \phi(\xi) d\xi$ --(19)

nd $I_2 = \frac{1}{W}\int_0^a \phi_2(\xi)\, k_o^2 \delta\varepsilon(\xi)\, \phi(\xi) d\xi$ --(20)

f course, $\phi(\xi)$ in (19) and (20) is yet unknown since it is the field n the inhomogeneous slab. We propose the following iterative echnique to solve the equations (15)-(18) for the four unknowns A, B, and T :

1- As a first iteration we put $\phi(\xi)$ in (19) and (20) equal to $_1(\xi)+\phi_2(\xi)$, to calculate the two integrals I_1 and I_2. Then we solve the our equations (15)-(18) for the four unknowns A, B, R and T. Let A_o, $_o$, R_o and T_o be their values for this first iteration. Using the values $_o$ and B_o we calculate the first iteration $\phi^I(x)$ for the field $\phi(x)$ from he equation (12) using the substitution $\phi(\xi) = \phi_1(\xi)+\phi_2(\xi)$ in the ntegral in the right hand side of equation (12).

2- Substitute $\phi^I(\xi)$ for $\phi(\xi)$ in the integrals I_1 and I_2, then olve (15)-(18) to obtain four new values A_1, B_1, R_1 and T_1 for the our unknowns A, B, R and T. Using A_1, B_1 and $\phi^I(\xi)$ we calculate the econd iteration $\phi^{II}(x)$ for the field $\phi(x)$ from equation (12).

This procedure can be continued until convergence to the final alues for A, B, R and T is reached. Usually only a few iterations are ufficient for convergence. The correctness of the results are checked rom the condition for energy conservation:

$$\sqrt{\varepsilon_i} = \sqrt{\varepsilon_i}\,|R|^2 + \sqrt{\varepsilon_t}\,|T|^2 \quad \text{------------------------(21)}$$

The terms $\sqrt{\varepsilon_i}$, $\sqrt{\varepsilon_i}|R|^2$ and $\sqrt{\varepsilon_t}|T|^2$ are proportional to the powers associated with the incident, reflected and transmitted waves respectively.

III- Numerical Applications

To test the validity of our method, we considered a linear permittivity profile:

$$\varepsilon(x) = \varepsilon_i + (\varepsilon_t - \varepsilon_i)(x/a) \quad , \quad 0 \le x \le a \text{ ---------------------- } (22)$$

This profile was studied previously by Chen[12] using an integral equation formulation. It is also amenable to analytical treatment: the solution can be expressed in terms of Airy functions[17]. The amplitudes of the reflection and transmission coefficients |R| and |T| are calculated functions of the profile height ε_t when $\varepsilon_i=1$ and the ratio $X=(a/\lambda_o)=1$ (where λ_o is the free space wavelength chosen equal to 1μm.). Figure 2 shows the results of our method (continuous curve) and Chen's results[12] (broken curve). A slight discrepancy between the two results occurs when the profile's height ε_t increases.

As a second test, we considered a linear profile with $\varepsilon_i=1$ and $\varepsilon_t=5$, and the following sinusoidal profile:

$$\varepsilon(x) = \varepsilon_i \; [\; 1 + \varepsilon_i(h-1)\sin(\pi x/a)] \quad \text{------------------ } (23)$$

Both of these profiles, the linear and the sinusoidal, were studied previously by Chen and Lien[11] using the finite element method. Figures 3 and 4 show the variations of |R| and |T| as functions of the ratio $X=a/\lambda_o$ (the normalized slab thickness), where $\lambda_o=1\mu$m. The results of our method (continuous curves) are almost identical to those obtained by the finite element method (broken curves).

V-Discussions

Chen and Lien[11] stated that Chen's integral equation :ormulation[12] has the disadvantage of dealing with a full matrix :roblem (which is usually associated with a relatively large error, :specially when large profile variations are to be considered). This is :hy they developed the finite element method to keep the error small. .lthough [11] was published after [12], we do not know why Chen and .ien did not make a comparative study between the finite element ethod[11] and the integral equation formulation developed by Chen.[12] comparative study between our integral equation and the finite :lement method is presented in this paper.

From figures 2, 3 and 4 we conclude that the agreement between our nethod and the finite element method is better than that between our nethod and Chen's[12] integral equation formulation.This led us to :hink that our formulation is better than Chen's one[12] because, as ›ointed out in [11], the finite element method is more accurate than :hen's integral equation formulation[12]. An important advantage of our :ormulation is that the kernel of the integral equation is of the :on-convolution type, this greatly simplifies the numerical integration :o be calculated in equation (12). Chen et. al.[11] have pointed out :hat when the finite difference method is used to treat such a problem :he results are very poor and not accurate and that is why they have :eveloped the finite element method.

- Conclusions

An integral equation formulation is presented for the problem of :ave propagation in an inhomogeneous dielectric slab. We applied a simple iterative technique to solve that equation numerically. To test our method, three previously studied permittivity profiles were considered. The results of our method agree with those of another integral equation formulation as shown in figure 2, but better agreement is achieved with the finite element method. The ability to deal with large profile variations was demonstrated when we considered steep linear and sinusoidal profiles. The results obtained by our method and the most accurate one (the finite element method) are almost identical as shown in figures 3 and 4.

Figure Captions

Figure 1 - A plane wave incident on an inhomogeneous dielectric slab occupying the space $0 \le x \le a$, and bounded by two homogeneous media having relative permittivities ε_i and ε_t.

Figure 2 - Amplitude of transmission and reflection coefficients $|T|$ and $|R|$ as function of profile height ε_t when $a=\lambda_o=1\mu m.$, $\varepsilon_i=1$

Figure 3 - Amplitude of the reflection coefficients $|R|$ as function of the normalized slab thickness $X=a/\lambda_o$ when $\varepsilon_i=\varepsilon_t=1$ and h=5. The letters S and L are for sinusoidal and linear profile respectively.

Figure 4- Amplitude of the transmission coefficients $|T|$ as function of the normalized slab thickness $X=a/\lambda_o$ when $\varepsilon_i=\varepsilon_t=1$ and h=5. The letters S and L are for sinusoidal and linear profile respectively.

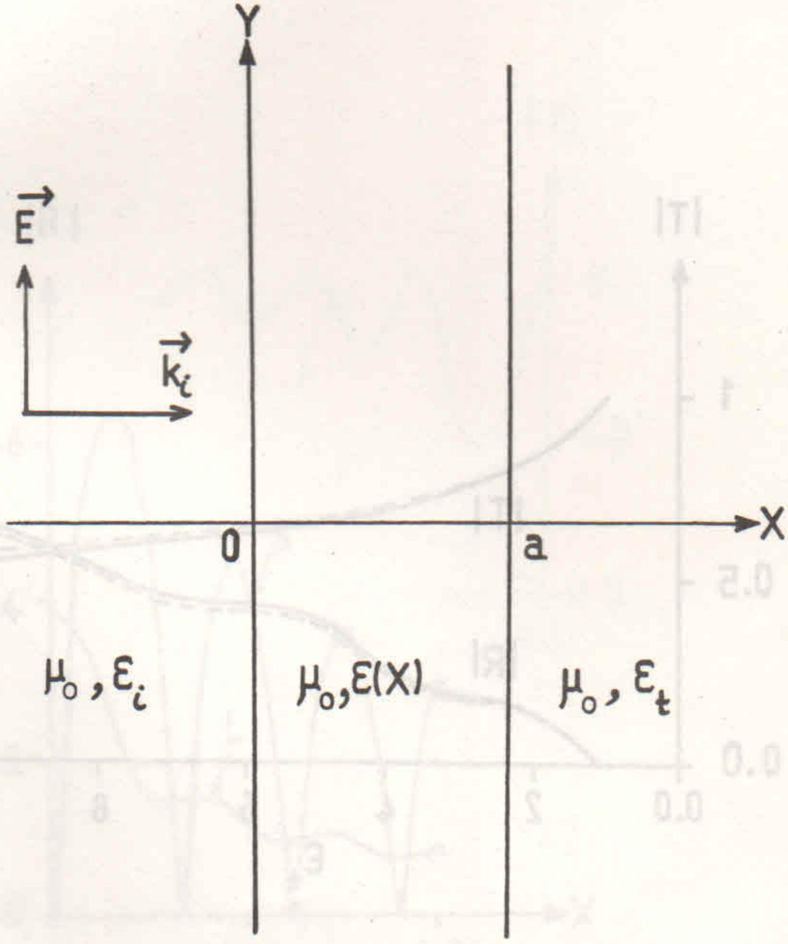
Y
$\vec{E}$
$\vec{k_i}$
0
a
X
μ_o , ε_i
$\mu_o, \varepsilon(X)$
μ_o , ε_t

Fig. 1

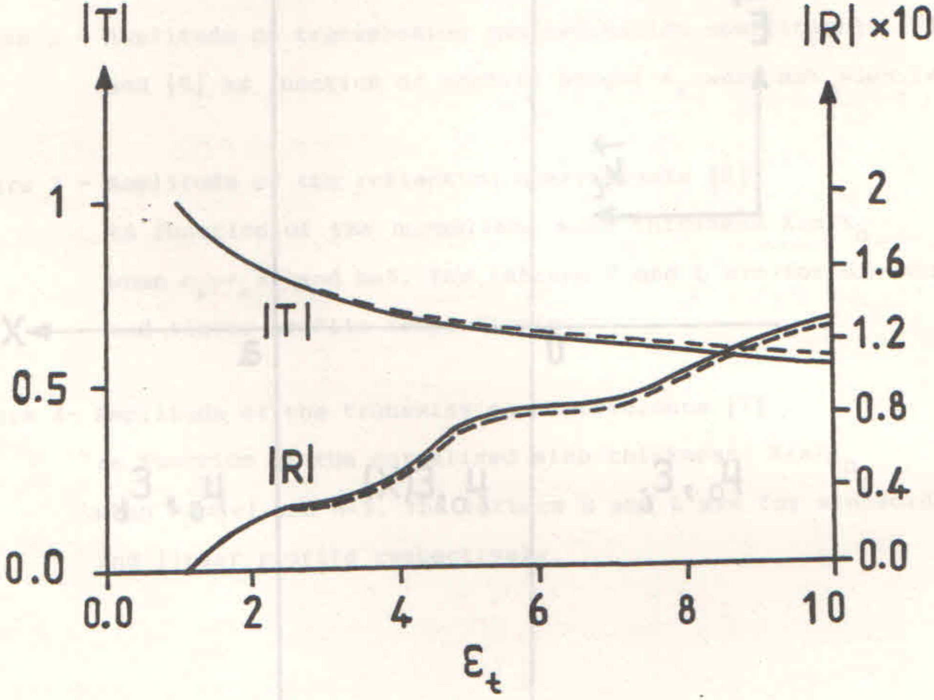

Fig. 2

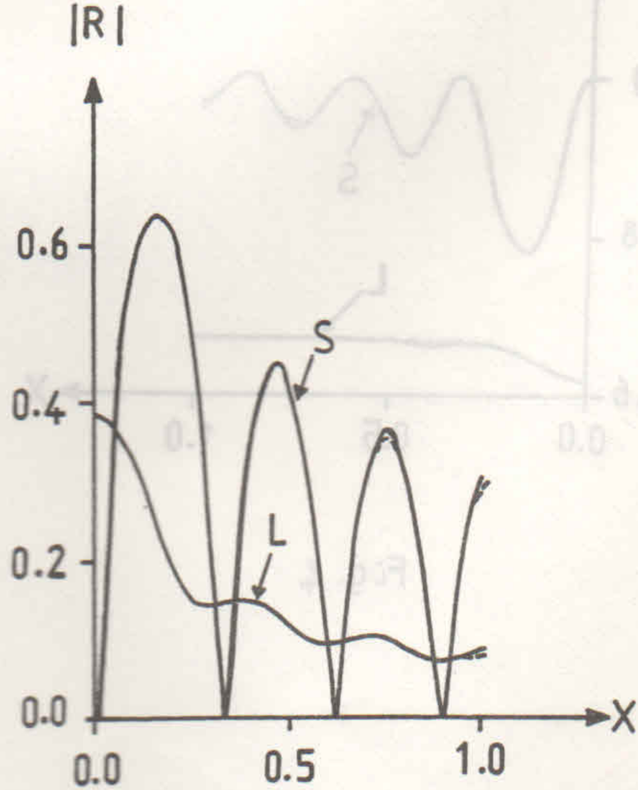

Fig. 3

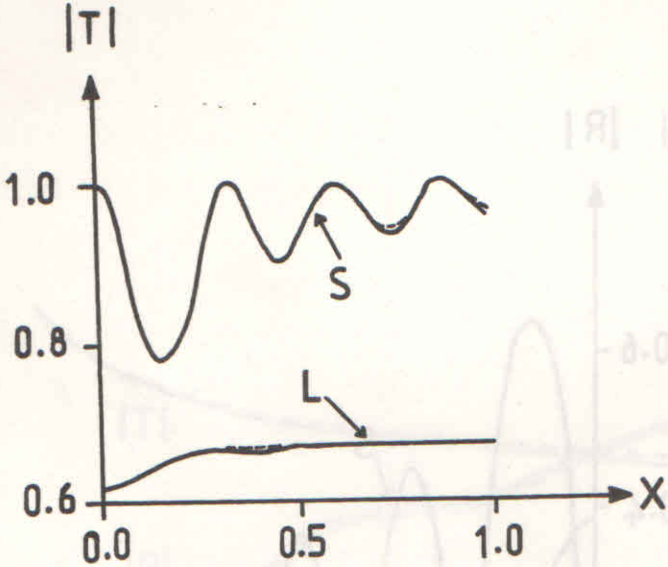

Fig. 4